Assignment 2 - Solutions

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1. PROBLEMS

1. Show that a stationary AR(1) process is ergodic for first and second moments. Discuss the implications of this fact on the estimation of the first and second population moments.
Consider the following stationary AR(1) process:

$$y_t = c + \phi y_{t-1} + \epsilon_t$$

where $|\phi| < 1$ and ϵ_t is white noise. Notice that a process is ergodic for the mean and the second moments if, respectively, the following conditions are satisfied:

- **mean:** $\gamma(j) \to 0$ for j that goes to ∞.
- **second moments:** $\sum_j^\infty |\gamma(j)| < \infty$

Clearly, as $\gamma(j) = \frac{\phi^j}{1-\phi^2}$ converges to 0 as j goes to infinity and its sum is smaller than infinity, we conclude that the stationary AR(1) is both ergodic for the mean and for the second moments. What does this mean in terms of estimation of the population moments? If a process is ergodic for the mean then the time average $\bar{y} = \frac{1}{T}\sum_{t=1}^T y_t \to E(y_t)$ as $T \to \infty$. This means that the sample average will converge in probability to the population mean as the sample size goes to infinity. Similarly, if a process is ergodic for the second moments, then:

$$\hat{\gamma}(j) = (1/T - j) \sum_{t=j+1}^{T} (y_t - \bar{y})(y_{t-j} - \bar{y}) \to \gamma(j)$$

for all j, for $T \to \infty$.

2. Consider the process X_t and assume it is weakly stationary with $E(X_t) = 3$.

i) Write a model for this process and describe its main characteristics.
Consider the following stationary model:

$$X_t = 3 + \Psi(L)\epsilon_t = 3 + \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$$

where ϵ_t is a white noise process $WN(0, \sigma^2)$ and ψ_j are absolutely summable.

ii) Compute its autocorrelation function.

Consider first the autocovariances of the process described above:

$$\gamma(1) = E((X_t - E(X_t))(X_{t-1} - E(X_{t-1}))) = \psi_0\psi_1\sigma^2 + \psi_1\psi_2\sigma^2 + \psi_2\psi_3\sigma^2 + \cdots = \sigma^2 \sum_{j=0}^{\infty} \psi_j\psi_{j+1}$$

$$\gamma(2) = E((X_t - E(X_t))(X_{t-2} - E(X_{t-2}))) = \psi_0\psi_2\sigma^2 + \psi_1\psi_3\sigma^2 + \psi_2\psi_4\sigma^2 + \cdots = \sigma^2 \sum_{j=0}^{\infty} \psi_j\psi_{j+2}$$

$$\gamma(k) = E((X_t - E(X_t))(X_{t-k} - E(X_{t-k}))) = \sigma^2 \sum_{j=0}^{\infty} \psi_j\psi_{j+k}$$

The autocorrelation is then given by:

$$\rho(k) = \frac{\gamma(k)}{var(X_t)} = \frac{\sum_{j=0}^{\infty} \psi_j\psi_{j+k}}{\sum_{j=0}^{\infty} \psi_j^2}$$

iii) Can you write X_t as an AR(∞) process? State the conditions you would need to impose so that your answer is affirmative.
If the process above is invertible, i.e. if the roots of the polynomial $\Psi(z)$ are bigger than one in absolute value, then we can rewrite it in terms of an AR(∞) process, as follows:

$$X_t = 3 + \Psi(L)\epsilon_t \quad \rightarrow \quad \Psi(L)^{-1}X_t = \Psi(L)^{-1}3 + \epsilon_t$$

How do we invert the polynomial $\Psi(L)$? We are looking for the values of the coefficients α_i of $(\alpha_0 + \alpha_1 L + \alpha_2 L^2 + \cdots)$ such that:

$$\Psi(L)\left(\alpha_0 + \alpha_1 L + \alpha_2 L^2 + \cdots\right) = 1$$

Suppose, for simplicity, that $\psi_0 = 1$. As all the coefficients of the non-zero powers of L must be equal to zero:

$$\alpha_0 = 1$$
$$\alpha_1 + \psi_1\alpha_0 = 0 \quad \Longrightarrow \quad \alpha_1 = -\psi_1$$
$$\alpha_2 + \psi_1\alpha_1 + \psi_2\alpha_0 = 0 \quad \Longrightarrow \quad \alpha_2 = \psi_1^2 - \psi_2$$
$$\alpha_3 + \psi_2\alpha_1 + \psi_1\alpha_2 + \psi_3\alpha_0 = 0 \quad \Longrightarrow \quad \alpha_3 = -\psi_1(\psi_1^2 - \psi_2) + \psi_2\psi_1 - \psi_3$$

so that the AR(∞) representation of the process is given by:

$$(1 + \sum_{i=1}^{\infty} \alpha_i L^i)X_t = \mu + \epsilon_t$$

where $\mu = (1 + \sum_{i=1}^{\infty} \alpha_i L^i)c$ and α_i are computed as above.

3. Consider the following processes. The sequence $\{\varepsilon_t\}$ is white noise in all cases.

a. $x_t = \mu + \varepsilon_t - \varepsilon_{t-1}$

b. $y_t = y_{t-1} + \varepsilon_t - 0.95\varepsilon_{t-1}$

c. $z_t = z_{t-1} + (1 - 0.5L)(1 - L)\varepsilon_t$

d. $(1 - L + 0.24L^2)v_t = (1 - 0.6L)\varepsilon_t$

i) State which of the above processes are ARMA(p,q) and for those processes state the values of p and q. (Note: Remember that ARMA processes are stationary. Note for c.: if a process has a unit root both in its MA and AR components, you can simplify this term from both sides of the equation).

- a) This process is an ARMA(0,1).
- b) Since the coefficient of the AR is equal to 1 in absolute value, the process is not stationary and therefore not an ARMA process.
- c) Notice that this process can be rewritten as follows:

$$(1 - L)z_t = (1 - 0.5L)(1 - L)\epsilon_t \quad \Longrightarrow \quad z_t = (1 - 0.5L)\epsilon_t$$

  Therefore, this is an ARMA(0,1).
- Notice that, since $|\theta| < 1$, the polynomial $(1 - 0.6L)$ is invertible and we can rewrite the process as follows:

$$(1 - 0.6L)^{-1}(1 - L + 0.24L^2)v_t = \epsilon_t \quad \Longrightarrow \quad (1 - 0.4L)v_t = \epsilon_t$$

  Therefore, this is an ARMA(1,0).

ii) Whenever possible write the processes above as AR processes.

- a) Notice that, as $|\theta| = 1$, the process is not invertible, therefore we can't write this process as an AR process.
- b) Notice that, since $|\theta| < 1$, the polynomial $(1 - 0.95L)$ is invertible and we can rewrite the process as follows:

$$(1 - 0.95L)^{-1}(1 - L)y_t = \epsilon_t \quad \Longrightarrow \quad (1 + 0.95L + (0.95)^2L^2 + \cdots)y_t = \epsilon_t$$

- c) In the previous point (i), we simplified the process as follows: $z_t = (1 - 0.5L)\epsilon_t$. As $|\theta| < 1$, we can invert the polynomial and rewrite the process in terms of an AR process:

$$(1 - 0.5L)^{-1}z_t = \epsilon_t \quad \Longrightarrow \quad (1 + 0.5L + (0.5)^2L^2 + \cdots)z_t = \epsilon_t$$

- b) Again, since $|\theta| < 1$, the polynomial $(1 - 0.6L)$ is invertible and we can rewrite the process as follows:

$$(1 - 0.6L)^{-1}(1 - L + 0.24L^2)v_t = \epsilon_t \quad \Longrightarrow \quad (1 - 0.4L)v_t = \epsilon_t$$

4. You are interested in estimating ϕ in the following regression model: $y_t = \phi y_{t-1} + \varepsilon_t$. Estimation is carried out using OLS.

a) Assuming that $|\phi| < 1$, discuss the asymptotic properties of $\hat{\phi}_{ols}$ in the following cases.

i. $\{\varepsilon_t\}$ is $i.i.d.N(0, \sigma^2)$
Recall that the OLS estimator can be written as follows:

$$\hat{\phi}_{ols} = \phi + \frac{\sum_{t=1}^{T} y_{t-1}\epsilon_t}{\sum_{t=1}^{T} y_{t-1}^2}$$

Provided that $\frac{\sum_{t=1}^{T} y_{t-1}\epsilon_t}{\sum_{t=1}^{T} y_{t-1}^2} \rightarrow 0$ as $T \rightarrow \infty$, we have that $\hat{\phi}_{ols}$ is a consistent estimator of ϕ. If $\epsilon_t \sim i.i.dN(0, \sigma^2)$, then $\hat{\phi}_{ols} \rightarrow \phi$, as $T \rightarrow \infty$ as $\sum_{t=1}^{T} y_{t-1}\epsilon_t \rightarrow E(y_{t-1}\epsilon_t) = 0$ (since ϵ_{t-1} is uncorrelated to ϵ_t), and it's asymptotically normal.

ii. $\varepsilon_t = \delta\varepsilon_{t-1} + v_t$, where v_t is a martingale difference sequence.
In this case, however:

$$\sum_{t=1}^{T} y_{t-1}\epsilon_t \rightarrow E(y_{t-1}\epsilon_t) = E(y_{t-1}(\delta\varepsilon_{t-1} + v_t)) = \delta E(\epsilon_{t-1}^2)$$

Provided that $\delta \neq 0$, $\hat{\phi}_{ols}$ will not be a consistent estimator of ϕ, even if $E(y_{t-1}v_t) = 0$.

b) Assuming that ε_t is defined as in ii), rewrite y_t as an AR(2) process with white noise residuals (v_t) and find the values of the AR coefficients as functions of ϕ and δ. Suppose that you estimate the resulting process by OLS. Is the estimator consistent in this case?
If ε_t is defined as in ii), then:

$$(1 - \delta L)\epsilon_t = v_t$$

Hence, the correctly specified model is:

$$y_t = \phi y_{t-1} + \frac{v_t}{(1 - \delta L)} \implies (1-\delta L)(1-\phi L)y_t = v_t \implies y_t = (\delta+\phi)y_{t-1} - \delta\phi y_{t-2} + v_t$$

Estimating the process by OLS, we obtain consistent estimates, as v_t is white noise and it is uncorrelated with the regressors y_{t-1} and y_{t-2}.

c) Consider now $y_t = \beta x_t + \varepsilon_t$ where ε_t is defined as in ii). Assuming that x_t and y_t are stationary and $cov(x_t, \varepsilon_t) = 0$, is the OLS estimator of β consistent in this case?
In this case, the OLS estimator would be consistent as:

$$\frac{\sum_{t=1}^{T} x_t\epsilon_t}{\sum_{t=1}^{T} x_t^2} \rightarrow E(x_t\epsilon_t) = 0$$

as $T \rightarrow \infty$.

5. A research wants to test hypotheses on ϕ, the AR(1) parameter from the model in exercise 3.i) . To that effect she constructs the t-statistic associated to the hypothesis $H_0 : \phi = 0$ and $H_0 : \phi = 1$. In both cases, her decision rule is as follows: she rejects the corresponding null hypothesis if the absolute value of the t-test is larger that the corresponding critical value obtained from the N(0,1). Discuss whether this procedure is right.
Provided that the sample size is large (remember that the t-student converges to a normal as the sample size goes to infinity), using the critical values from the N(0.1) to assess whether

$H_0 : \phi = 0$ is rejected or not is right. In the case in which we want to assess whether there is a unit root or not, i.e. we test the null $H_0 : \phi = 1$, then we can't use the critical values from the normal distribution. Why is this the case? As you've seen in class, the validity of LLNs and CLTs relied on stationarity+ergodicity. Processes with unit roots are not stationary, so these do not hold. The asymptotic distribution of the autoregressive parameter is not normal, but a functional of Brownian motions. These distributions are non-standard and require specific tabulation. Using critical values from standard distributions is therefore wrong.

2. Computer practice

6. Using MATLAB, construct a function that for a given data, i) selects the order k using the General-to-Specific selection method corresponding to an AR specification and ii) estimates the corresponding AR(k) process using OLS. The input of the function should be your data and the maximum lag for the AR. The output should be k and the estimates and variances of the estimates.

7. For the two variables accompanying this problem set i) identify an appropiate ARMA model for (some transformation that "looks" stationary of) your data using the AIC and the BIC as well as visual inspection of the data and its ACF and PACF, 2) estimate the model you think is most suitable for your data and 3) check the residuals to see whether the model satisfies the main identification assumptions.