Topics in Applied Econometrics for Public Policy

Master in Economics of Public Policy, BSE

Handout 3: Nonparametric Series Regression

Laura Mayoral

IAE and BSE

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1. Introduction

- Previous handout: nonparametric local weighted average estimators.

- Several methods: NW (local constant), local linear, local polynomial Lowess, K-NN, etc.

- Recall these are local average methods, (i.e., averages of the dependent variable) where the weights employed are Kernel weights.

- Now: a new class of nonparametric regression methods: nonparametric series regression.

- Goal: Same as before, estimate the conditional expectation.

Series Regression

- Model: Consider two random variables (y, x) who are related by,

$$y = m(x) + e \tag{1}$$

where $\mathbb{E}[e|x] = 0$ and $\mathbb{E}[e^2|x] = \sigma^2(x)$.

- Goal: estimate m(.), unspecified

- Idea: approximate m(x) by a flexible function.

- We focus on linear functions (other possibilities also exist but linear functions are simple and work well)

- In particular
  - polynomials
  - splines

Series Regression, II

- Linear series regression models take the form

$$y = X_K' \beta_K + e_K \tag{2}$$

- where X_K is a vector of regressors obtained by transforming x in different ways
- β_K is a coefficient vector.

- We examine next two popular series regression estimators
- Polynomials
- Splines

Polynomial Regression

- Conditional expectation:

- approximated by a polynomial in x of degree p:

$$m_K(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_p x^p$$

.

- number of parameters to be estimated is $K = p + 1$

- Simple approach: estimate b_k by OLS

- p: controls the degree of flexibility of a polynomial regression. Tradeoff
  - A large p provides a lot of flexibility
  - But it can become too noisy

Example

- (from Hansen's book, chapter 20)
- Log wages on experience for women with college education (education= 16), separately for white women and Black women

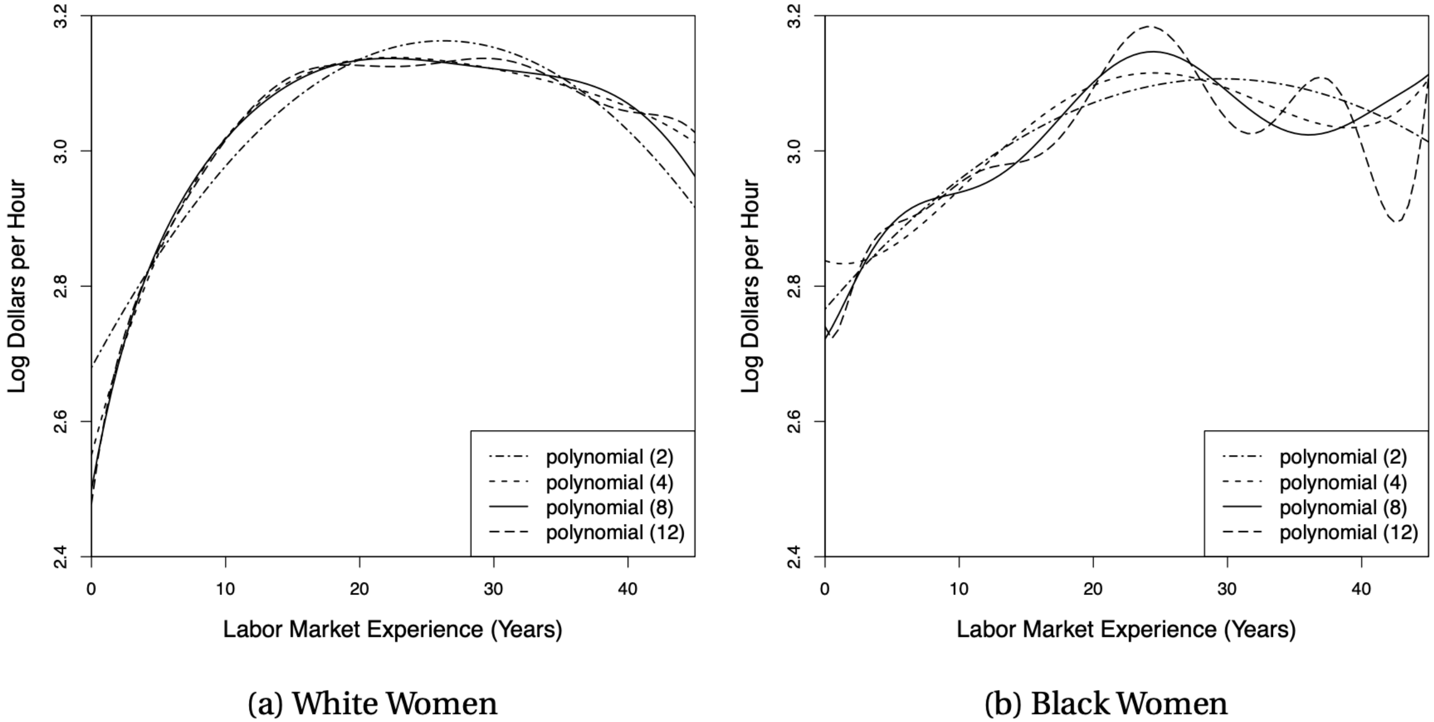


Figure 20.1: Polynomial Estimates of Experience Profile, College-Educated Women

- Difference between the two plots: might be due to the fact that the sub-sample of Black women has much fewer observations

- Then, the mean function is much less precisely estimated, giving rise to the erratic plots

Orthogonal polynomials

- The different regressors $(x^1, x^2, \ldots, x^j \ldots)$ can be highly correlated

- Then the OLS estimator can be difficult to compute (as it needs to invert a near-singular matrix)

- One solution: orthogonalize the polinomial.

- Goal of orthogonal polynomials: get rid of the problem of the inversion of $X_k' X_k$

- How they work: they produce regressors that are close to being orthogonal and have similar variances, which implies that the resulting matrix of orthogonal regressors $X_k^{*\prime} X_k^*$ is diagonal and with similar diagonal values (the variances).

- Then, use this vector of orthogonal regressors rather than X_k.

- There exist different ways of doing these orthogalizations, for instance: 1) sample orthogonalization and) use orthogonal polynomials

- The most popular orthogonal polynomials are:

- Hermite polynomial, Laguerre Polynomial etc.

(See Hansen, chapter 20 for further details)

Implementation in STATA: npregress series

From STATA help:

- npregress series: performs nonparametric series estimation
- Like linear regression, nonparametric regression models the mean of the outcome conditional on covariates
- but unlike linear regression, it makes no assumptions about the functional form of the relationship between the outcome and the covariates.
- Output: average marginal effect

- log wages on years of education.
- stata command: npregress series lnhwage educatn, polynomial
- Output: average effect
- Polynomial order: chosen by cross-validation

```
Minimizing cross-validation criterion

Iteration 0:  Cross-validation criterion =  .6118535
Iteration 1:  Cross-validation criterion =  .5714533

Computing average derivatives

Polynomial-series estimation                  Number of obs      =        177
Criterion: cross-validation                   Polynomial order   =          3
```

lnhwage	Effect	Robust std. err.	z	P>|z|	[95% conf.	interval]
educatn	.1488311	.0195814	7.60	0.000	.1104523	.1872099

Note: Effect estimates are averages of derivatives.

■ Different output if regressor is continuous or discrete. Education has 14 different values. Now we enter it in the model as discrete

npregress series lnhwage i.educatn, polynomial

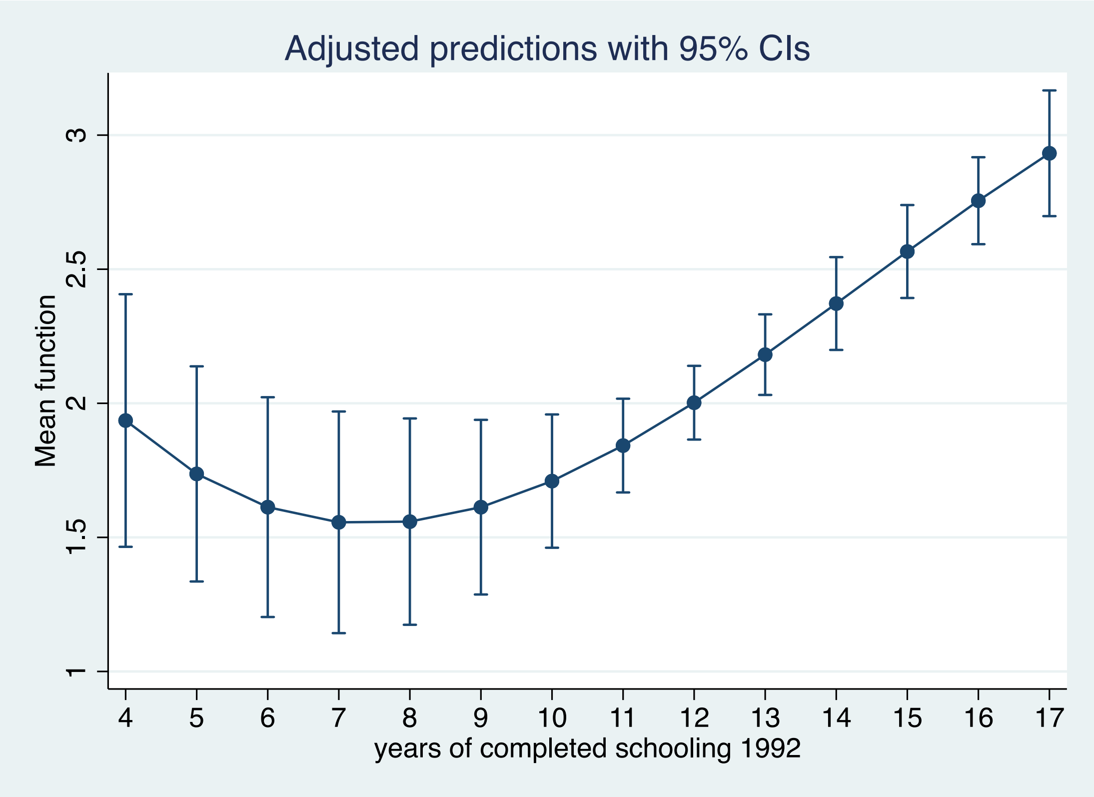
```
. npregress series lnhwage i.educatn, polynomial

Computing approximating function

Minimizing cross-validation criterion

Iteration 0:  Cross-validation criterion =  .6576989

Computing average derivatives

Polynomial-series estimation               Number of obs      =            177
Criterion: cross-validation                Polynomial order   =              1
```

lnhwage	Effect	Robust std. err.	z	P>|z|	[95% conf.	interval]
educatn						
(4 vs 3)	-.43711	.5819696	-0.75	0.453	-1.57775	.7035295
(6 vs 3)	.0336978	.504947	0.07	0.947	-.9559802	1.023376
(7 vs 3)	-.0246969	.4939832	-0.05	0.960	-.9928862	.9434923
(8 vs 3)	-1.022982	1.072799	-0.95	0.340	-3.125629	1.079666
(9 vs 3)	-.6525539	.6075096	-1.07	0.283	-1.843251	.5381431
(10 vs 3)	-.6353192	.5691324	-1.12	0.264	-1.750798	.4801598
(11 vs 3)	-.7832856	.6340793	-1.24	0.217	-2.026058	.459487
(12 vs 3)	-.0904436	.5003293	-0.18	0.857	-1.071071	.8901838
(13 vs 3)	-.0444978	.515248	-0.09	0.931	-1.054365	.9653696
(14 vs 3)	.3913532	.5309523	0.74	0.461	-.6492942	1.432001
(15 vs 3)	.0298562	.5860523	0.05	0.959	-1.118785	1.178498
(16 vs 3)	.5577673	.5102761	1.09	0.274	-.4423555	1.55789
(17 vs 3)	.853556	.5015454	1.70	0.089	-.1294549	1.836567

Note: Effect estimates are averages of contrasts of factor covariates.

Estimated function at different data points

npregress series lnhwage educatn, polynomial margins, at(educatn=(4 5 6 7 8 9 10 11 12 13 14 15 16 17)) marginsplot)



Splines

- A spline is a piecewise polynomial.

- Order of polynomial: pre-selected to be linear, quadratic, or cubic.

- The flexibility of the model: determined by the number of polynomial segments.

- The join points between the segments are called knots.

- If there's 1 knot, there are two segments, etc.

Splines, II

- How to construct a spline?

- Choose p (order of the polynomial), typically p=1, 2 or 3

  - A quadratic or cubic spline is useful when it is desired to impose smoothness

  - a linear spline is useful when it is desired to allow for sharp changes in slope.

- Choose number of knots

Examples

- Example 1: a linear spline with one knot τ:

(we allow the slope to change once)

$$m_K(x) = \beta_0 + \beta_1 x + \beta_2 (x - \tau) \mathbb{1}_{(x \geq \tau)}$$

Notice that;

- for $x < \tau$, $m_K(x) = \beta_0 + \beta_1 x$ is linear with slope β_1;
- for $x \geq \tau$, $m_K(x)$ is linear with slope $\beta_1 + \beta_2$; and the function is continuous at $x = \tau$.
- β_2 is the change in the slope at τ.

- Example 2: A linear spline with two knots $\tau_1 < \tau_2$:

(The knots allow the slope to change twice)

$$m_K(x) = \beta_0 + \beta_1 x + \beta_2 (x - \tau_1)\mathbb{1}(x \geq \tau_1) + \beta_3 (x - \tau_2)\mathbb{1}(x \geq \tau_2)$$

.

- Example 3: quadratic spline with one knot is (we allow the coefficient of x^2 to change once)

$$m_K(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 (x - \tau)^2 \cdot 1(x \geq \tau)$$

- In general, a pth-order spline with N knots $\tau_1 < \tau_2 < \cdots < \tau_N$ is

$$m_K(x) = \sum_{j=0}^{N+p-1} \beta_j x^j + \sum_{k=1}^{N} \beta_{p+k} (x - \tau_k)^p \cdot 1x \geq \tau_k$$

- Important: select the number and location of knots.
- As usual, many options for doing this
- Simplest: evenly spaced

Example 1 (Hansen, Chapter 20)

- Graph plots log wages on experience for Black women (394 obs.)

- quadratic spline (smooth changes)

- four equally-spaced knots at experience levels of 10, 20, 30, and 40 (7 coefficients)

- For comparison: 6th order polynomial regression (also 7 coefficients).

(a) Experience Profile

- Interpretation example 1:

- the spline is a quadratic over each segment, but the first two segments (experience levels between 0-10 and 10-20 years) are essentially linear.

- Most of the curvature occurs in the third and fourth segments (20-30 and 30-40 years) where the estimated regression function peaks and twists into a negative slope.

- The estimated regression function is smooth.

Example 2 (Hansen, Chapter 20)

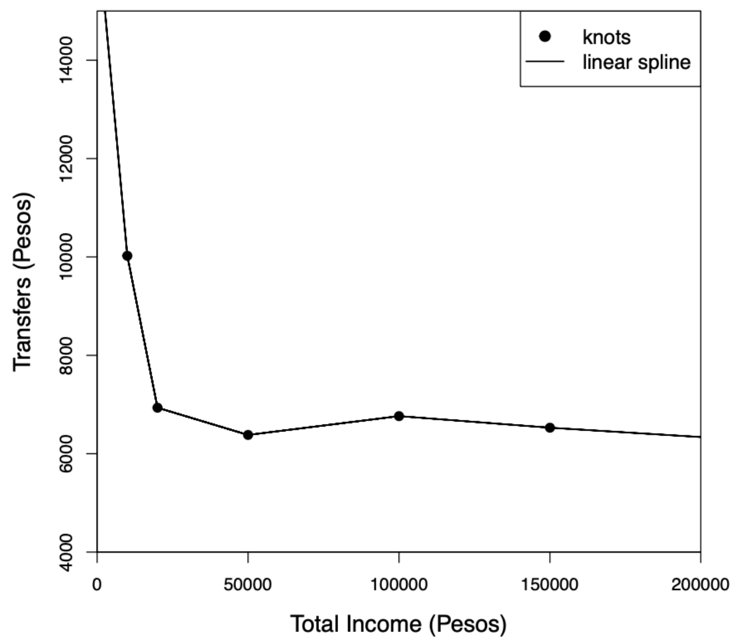
- A model of altruistic transfers: transfers of extended family. vs. income family.

- Model predicts that extended families will make gifts (transfers) when the recipient family's income is low, but will not make transfers if the recipient family's income exceeds a threshold.

- A pure altruistic model predicts that the regression of transfers received on family income should have a slope of 1 up to this threshold and be flat above this threshold.

(sharp changes)

- linear spline with knots at 10000, 20000, 50000, 100000, and 150000 pesos.



(b) Effect of Income on Transfers

Splines in STATA

npregress series lnhwage educatn, spline

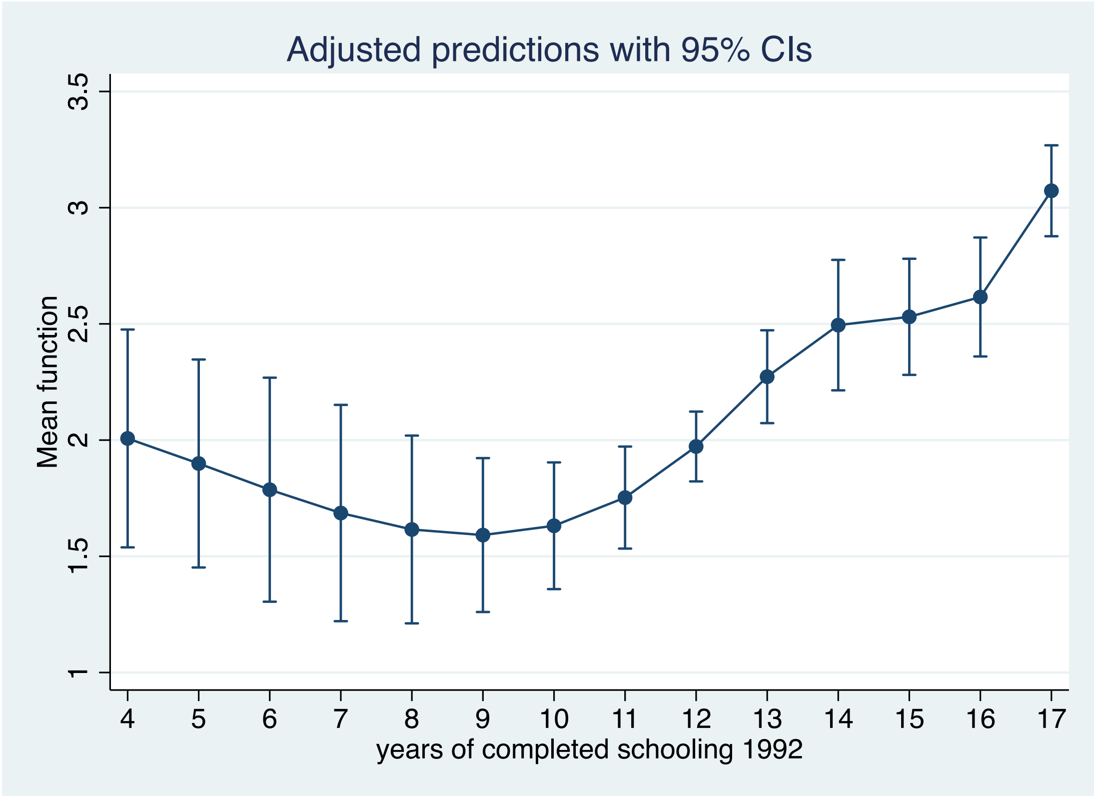
Note: unless specified otherwise, cubic spline and number of knots chosen by cross validation

In this example: cubic spline, 3 knots...how many parameters?

```
. npregress series lnhwage educatn, spline
warning: you have entered variable educatn as continuous but it only has 14 distinct values.  The e
        substantially if you inadvertently include a discrete variable as continuous

Computing approximating function

Minimizing cross-validation criterion

Iteration 0:  Cross-validation criterion =  .5835807
Iteration 1:  Cross-validation criterion =  .5803272

Computing average derivatives

Cubic-spline estimation                        Number of obs     =        177
Criterion: cross-validation                    Number of knots   =          3

------------------------------------------------------------------------------
             |               Robust
     lnhwage |     Effect   std. err.      z    P>|z|     [95% conf. interval]
-------------+----------------------------------------------------------------
     educatn |   .2100777   .0385008     5.46   0.000     .1346175    .2855378
------------------------------------------------------------------------------
Note: Effect estimates are averages of derivatives.
```

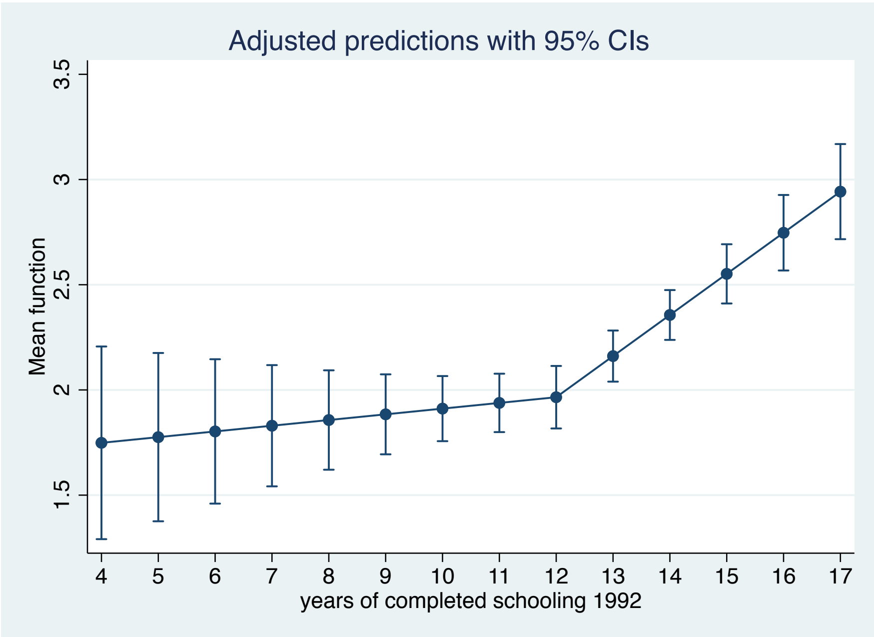
- Conditional expectation at different points

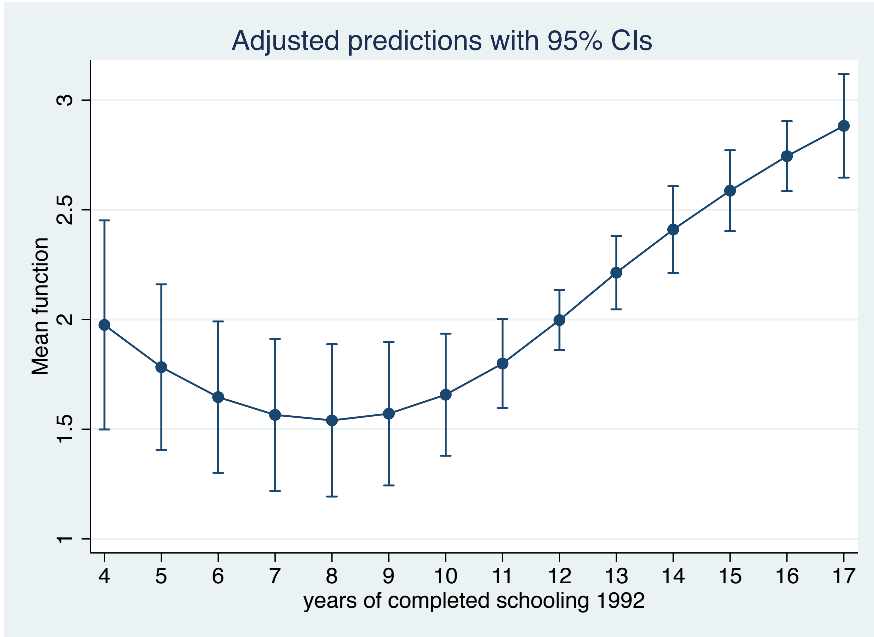
```
margins, at(educatn=(4 5 6 7 8 9 10 11 12 13 14 15 16 17))
marginsplot
```



- Restrict to linear or quadratic spline to see the difference

(in both cases, 1 knot selected by CV)





Asymptotic properties

- Consistent (if $K, N \to \infty$)
- Asymptotically normal
- Rate of convergence is $\sqrt{N}$
- But finite-samples biases still exist (the C.I. is not centered correctly centered), similar case as in Kernel regression!
- the bias term can be made asymptotically negligible if we assume that

K increases with N at a sufficiently fast rate.

The Global/Local Nature of Series Regression

- Kernel regression as inherently local in nature.

- The Nadaraya-Watson, Local Linear, and Local Polynomial estimators estimate $m(x_0)$ only considering x's close to x_0.

- In contrast, series regression is typically described as global in nature: estimators are a function of the whole sample

- However, series regression estimators share the local smoothing property of kernel regression:

- As the number of series terms K increases a series estimator also becomes a local weighted average estimator.

- Thus, another interpretation

  - Both kernel and series regression are global in nature when h is large (kernels) or K is small (series), and

  - . . . are local in nature when h is small (kernels) or K is large (series).

- See Hansen, Chapter 20, for additional details

Takeaways

- Two different ways of doing nonparametric regression
- Local averages
- This handout: Flexible functions of the regressors:
  - Splines
  - Polynomials
- Very easy to implement, quicker rate of convergence

Additional References

In case you are interested in this topic, you can check the following references:

- For a textbook treatment of series regression: see Li and Racine (2007).

- For an advanced treatment see Chen (2007).

- Two seminal contributions are Andrews (1991a) and Newey (1997).

- Recent contributions: Belloni, Chernozhukov, Chetverikov, and Kato (2015) and Chen and Christensen (2015).